Directed and multi-directed animals on the King's lattice

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Abstract. We define the *directed King's lattice* to be the square lattice with diagonal (next nearest neighbor) bonds and with the preferred directions $\{\leftarrow,\nwarrow,\uparrow,\nearrow,\rightarrow\}$. We enumerate directed animals on this lattice using a bijection with Viennot's heaps of pieces. We also define and enumerate a superclass of directed animals, the elements of which are called multi-directed animals. This follows Bousquet-Mélou and Rechnitzer's work on the directed triangular and square lattices. Our final results show that directed and multi-directed animals asymptotically behave similarly to the ones on the triangular and square lattices.

1 Introduction

An animal on a lattice is a finite and connected set of vertices. The enumeration of animals (up to a translation) is a longstanding problem in statistical physics and combinatorics. The problem, however, is extremely difficult, and little progress has been made [16, 12]. A more realistic goal, therefore, is to enumerate natural subclasses of animals.

The class of *directed animals* is one of the most classical of these subclasses. Directed animals have been enumerated in a variety of lattices; let us cite, non-exhaustively, the square and triangular lattices [18, 14, 9, 11, 2], Bousquet-Mélou and Conway's lattices $\mathscr{L}_n$ [4, 8], and the "strange" or n-decorated lattices [7, 3] (Figure 1). Unsolved lattices include, notably, the honeycomb lattice [13].

The class of *multi-directed animals* is a superclass of directed animals, first introduced by Klarner [15] on the square and triangular lattices. Bousquet-Mélou and Rechnitzer [5] clarified Klarner's definition and introduced a variant class on the square lattice. Moreover, they gave closed expressions for the generating function of multi-directed animals and showed that it is not D-finite.

The goal of this paper is to enumerate directed and multi-directed animals on a new lattice. We call *King's lattice* the square lattice with added diagonal bonds, or next nearest neighbor bonds. We also consider the preferred orientations $\{\leftarrow,\nwarrow,\uparrow,\nearrow,\rightarrow\}$ (Figure 1, right). Directed animals on the King's lattice are a superclass of directed animals on Bousquet-Mélou and Conway's lattice $\mathscr{L}_3$, which has arcs $\{\nwarrow,\uparrow,\nearrow\}$ [4].

Several techniques have been used to enumerate directed animals on the various lattices. Among them are direct bijections with other combinatorial objects [11], comparison with gas models [9, 3, 17, 1] and the use of Viennot's theory of heaps of pieces [20, 2, 8, 5, 21]. In this paper, we use the last method; we show that directed animals on the King's lattice are in bijection with heaps of segments, already defined in [6].

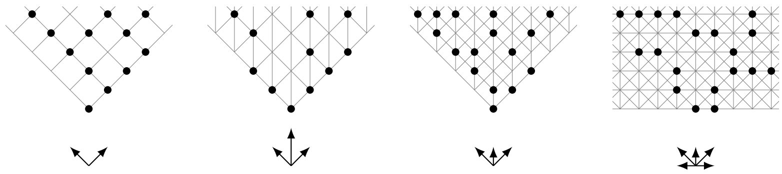

Fig. 1. Directed animals on a selection of lattices. From left to right: the square lattice, the triangular lattice, the lattice $\mathscr{L}_3$, and the King's lattice.

2 Animals on the King's lattice and heaps of segments

2.1 Definitions

Definition 1. We call *segment* a closed real interval of the form $[i, j]$, where i and j are integers such that $j > i$. Two segments are called *concurrent* if they intersect, even by a point. A *heap of segments* is a finite sequence of segments, considered up to commutation of non-concurrent segments.

The heaps of segments described here are the same as in [6], except that the segment reduced to a point is not allowed. More information on heaps of pieces in general can be found in [20]. Graphically, a heap is built by dropping segments in succession; a segment either falls on the ground or on another segment concurrent to it. Examples are shown in Figures 2 and 3.

2.2 Directed animals and pyramids of segments

Let A be an animal; we say that a site t of A is *connected* to another site s if there exists a directed path (*i.e.* respecting the preferred directions of the lattice) from s to t visiting only sites of A. We say that the animal A is *directed* of source s if every site t of A is connected to s. The source s is not unique; it may be any of the bottommost sites of A (see Figure 2, left). By convention, we call *source* of A the leftmost bottom site.

In Figure 2 is illustrated a bijection between directed animals and *pyramids of segments* (or heaps with only one segment lying on the ground). This bijection works identically to the classical bijection between directed animals on the square lattice and strict pyramids of dimers [20, 2].

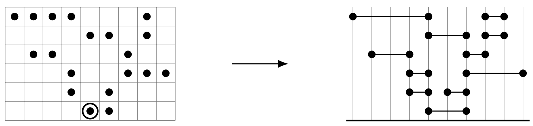

Fig. 2. Left: a directed animal on the King's lattice (represented, for clarity, as a polyomino on the dual lattice) with its source circled. Right: the pyramid of segments obtained by replacing each maximal sequence of ℓ consecutive sites by a segment of length ℓ.

2.3 Multi-directed animals and connected heaps of segments

Let A be an animal. For any abscissa i, we denote by $b(i)$ the ordinate of the bottommost site of A at abscissa i (or $b(i) = +\infty$ if there is no site of A at abscissa i). We call *source* of A a site that realizes a local minimum of b and *keystone* of A

a site that realizes a local maximum. In case several consecutive sites realize a minimum or maximum, the source or keystone is the leftmost one (Figure 3, left). This is a purely arbitrary choice that does not alter the definition.

Definition 2. Let A be an animal. The animal A is said *multi-directed* if it satisfies the two conditions:

– for every site t of A, there exists a source s such that t is connected to s;
– for every keystone t of A, there exist two sources s_ℓ and s_r, to the left and to the right of t respectively, such that t is connected to both s_ℓ and s_r. Moreover, the directed paths connecting t to s_ℓ and s_r do not go through a keystone at the same height as t.

As a directed animal has only one source and no keystone, every directed animal is multi-directed. Multi-directed animals are in bijection with *connected heaps of segments* (or heaps without an empty column). A multi-directed animal and its corresponding heap are depicted in Figure 3.

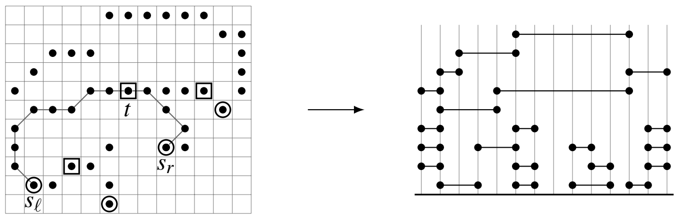

Fig. 3. Left: a multi-directed animal with four sources (circled) and three keystones (boxed). The directed paths connecting one keystone, denoted by t, to the sources s_ℓ and s_r are shown. Right: the corresponding connected heap of segments, with has four minimal pieces (one for each source of the animal).

Definition 2 can be adapted in the directed square and triangular lattices; the animals thus defined are in bijection with *connected heaps of dimers*. Bousquet-Mélou and Rechnitzer also defined multi-directed animals in bijection with connected heaps of dimers in [5], in a slightly different way. Our definition of multi-directed animals has the advantages of being more intrinsic and of having a vertical symmetry.

3 Enumeration of directed animals

In this section, we use the bijection with pyramids of segments to enumerate directed animals on the King's lattice. We call *half-animal* a directed animal with no site on the left side of its source. The associated pyramids are called *half-pyramids*. We adapt Bétréma and Penaud's methods [2] to decompose the pyramids of segments, which yields the following result.

Theorem 3. *The generating functions $S(t)$ and $D(t)$ of half-animals and animals are:*

$$S(t) = \frac{1-3t-\sqrt{1-6t+t^2}}{4t}; \qquad D(t) = \frac{1}{4}\left(\frac{1+t}{\sqrt{1-6t+t^2}} - 1\right).$$

The decomposition of the half-pyramids is sketched in Figure 4. Interestingly, the generating function $S(t)$ is already known in combinatorics. Its coefficients are the little Schröder numbers, **A001003** in the OEIS [19]. The coefficients of $D(t)$ also appear as **A047781**. This is remindful of the triangular lattice, where the half-animals are enumerated by the Catalan numbers [2].

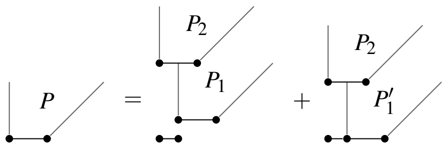

Fig. 4. Sketch of the two cases in the decomposition of half-pyramids. The generating function of the possible heaps P_1 and P_2 is $1+S(t)$, while the generating function of the possible heaps P_1' is $S(t)$. This shows the identity $S = t(1+S)^2 + tS(1+S)$, from which we derive the value of $S(t)$.

4 Enumeration of multi-directed animals

In this section, we enumerate multi-directed animals, or, equivalently, connected heaps of segments. To do this, we adapt the *Nordic decomposition*, invented by Viennot to enumerate connected heaps of dimers [21].

Theorem 4. *Let $M = M(t)$ be the generating function of multi-directed animals. Let $S = S(t)$, $D = D(t)$ be the power series defined in Theorem 3, $R = S + t(1+S)$ and $Q = 2(1-t)S - t$. The generating function M is given by:*

$$M = \frac{D}{1 - \sum_{k\geq 0} S(1+S)^k \frac{QR^k}{1-QR^k}}.$$

5 Asymptotic results

Finally, we derive asymptotic results from Theorems 3 and 4.

Theorem 5. *Let D_n and M_n be the number of directed and multi-directed animals of area n, respectively. As n tends to infinity, we have:*

$$D_n \sim \kappa\big(3+\sqrt{8}\big)^n n^{-1/2}; \qquad\qquad M_n \sim \lambda\mu^n,$$

with $\mu = 6.475\ldots$. The average width of directed animals grows like $\sqrt{n}$, while the average width of multi-directed animals grows like n. Finally, the series $M(t)$ is not D-finite.

The results on directed animals are a straightforward application of singularity analysis [10, Theorem VI.4]. The results on multi-directed animals are more involved. Similar results already exist on the square and triangular lattices, including the non-D-finiteness of the series $M(t)$ [5].

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